RIEMANNIAN FOLIATIONS AND THE TOPOLOGY OF LORENTZIAN MANIFOLDS

KORDIAN LÄRZ

Abstract. A parallel lightlike vector field on a Lorentzian manifold X naturally defines a foliation $\mathcal{F}$ of codimension one. If either all leaves of $\mathcal{F}$ are compact or X itself is compact admitting a compact leaf and the (transverse) Ricci curvature is non-negative then a Bochner type argument implies that the first Betti number of X is bounded by $1 \leq b_1 \leq \dim X$ if X is compact and $0 \leq b_1 \leq \dim X - 1$ otherwise. We show that these bounds are optimal and depending on the holonomy of X we obtain further results. Finally, we classify the holonomy representations for those X admitting a compact leaf with finite fundamental group.

1. The Class of Decent Spacetimes

Let (X, g^L) be a Lorentzian manifold and ∇^L its Levi-Civita connection.[1] Suppose (X, g^L) admits a ∇^L-parallel lightlike subbundle $\Xi \subset TX$ of rank one, i.e., $\nabla^L \Gamma(U \subset X, \Xi) \subset \Gamma(U, \Xi)$. We write $\Xi^\perp \subset TX$ for its orthogonal complement. Thus, $\Xi^\perp \supset \Xi$ has codimension one. Being a ∇^L-parallel subbundle, Ξ and therefore $\Xi^\perp$ induce a foliation $\mathcal{X}$ of dimension one and a foliation $\mathcal{X}^\perp$ of codimension one on X. Consider the vector bundle $\mathcal{S} := \mathrm{Coker}(\Xi \hookrightarrow \Xi^\perp)$. We have an induced metric $h^{\mathcal{S}}$ and an induced connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$. Moreover, $h^{\mathcal{S}}$ has Riemannian signature and $\nabla^{\mathcal{S}} h^{\mathcal{S}} = 0$. We call $(\mathcal{S}, h^{\mathcal{S}}, \nabla^{\mathcal{S}})$ the (canonical) screen bundle of (X, g). Given a non-canonical splitting s of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Xi \arrow[r] & \Xi^\perp \arrow[r] & \mathcal{S} \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$

we define $S := s(\mathcal{S})$ and call it a (non-canonical) realization of $\mathcal{S}$ in TX. The connection ∇^L on X induces a connection on S given by $\nabla^S := pr_S \circ \nabla^L|_S$.

The canonical bundle morphism $S \xrightarrow{F} \mathcal{S}$ is easily shown to be a vector bundle isomorphism such that $\nabla^S = F^* \nabla^{\mathcal{S}}$ and $g|_{S \times S} = F^* h^{\mathcal{S}}$, i.e., $Hol(S, \nabla^S) = Hol(\mathcal{S}, \nabla^{\mathcal{S}})$. Since $\Xi \subset S^\perp$ and $S^\perp \subset TX$ has signature $(1,1)$ the light cone in $S_p^\perp$ is the union of two lines one of which is given by Ξ_p and we derive

Corollary 1.1. *Given a realization $S \subset TX$ of the screen bundle of (X, g^L) there is a uniquely defined lightlike subbundle $\Theta \subset TX$ of rank one with the following property: If $V \in \Gamma(U \subset X, \Xi)$ then there exists a unique section $Z \in \Gamma(U \subset X, \Theta)$ such that $g^L(V, Z) = 1$.* ∎

Using locally future pointing sections as well as Cor. 1.1 and a partition of unity we conclude that the following are equivalent.

- Ξ admits a nowhere vanishing section,
- (X, g^L) is time-orientable,
- $\mathcal{X}^\perp$ is transversely orientable.

2000 *Mathematics Subject Classification.* 53C29, 53C50.

Key words and phrases. Lorentzian geometry, holonomy group, Riemannian foliation, pp-wave.

Acknowledgment: I am grateful to Helga Baum for many useful discussions on the topic.

[1]All manifolds are assumed to be connected, Lorentzian manifolds are assumed to be orientable.

Since Ξ is ∇^L-parallel any global section is recurrent.[2]

Definition 1.2. Let (X, g^L) be a Lorentzian manifold and $V \in \Gamma(X, TX)$ a global nowhere vanishing lightlike vector field. We say (X, g^L, V) is an

(1) almost decent spacetime if $\nabla^L_\cdot V = \alpha(\cdot)V$ for some 1-form $\alpha \in \Gamma(X, T^*X)$.
(2) decent spacetime if it is almost decent and $\alpha|_{\Xi^\perp} = 0$. ■

For an almost decent spacetime we always assume that $V \in \Gamma(X, \Xi)$ is future pointing. Next, we characterize the class of almost decent spacetimes in the class of Lorentzian manifolds. If (X, g^L) is an arbitrary Lorentzian manifold let $\mathfrak{hol}_p(X, g^L)$ be its holonomy algebra at $p \in X$. Then $\mathfrak{hol}_p(X, g^L)$ has the Borel-Lichnérowicz property, i.e., there is an orthogonal decomposition $T_pX = E_0 \oplus \ldots \oplus E_\ell$ into non-degenerate $\mathfrak{hol}_p(X, g^L)$-invariant subspaces and a corresponding decomposition $\mathfrak{hol}_p(X, g^L) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$ into commuting ideals such that each $\mathfrak{h}_j \subset \mathfrak{so}(E_j, g^L|_{E_j})$ acts weakly irreducibly on E_j and trivially on E_i for $i \neq j$. Using [DSO01] we derive three possible cases:

(1) $E_0 = 0$ or $g^L|_{E_0}$ is positive definite and $\mathfrak{h}_i$ acts irreducibly for $i \geq 1$. In this case, we may assume that $g^L|_{E_j}$ is positive definite for $j \geq 2$. Hence, $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 2$ and $\mathfrak{h}_1 = \mathfrak{so}(1, n+1)$.
(2) $E_0 \neq 0$ and $g^L|_{E_0}$ is negative definite or of Lorentzian signature. Thus, $g^L|_{E_j}$ is positive definite and $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 1$.
(3) $E_0 = 0$ or $g^L|_{E_0}$ is positive definite, $\mathfrak{h}_j$ acts as an irreducible Riemannian holonomy representation for $j \geq 2$ and $\mathfrak{h}_1 \subset \mathfrak{so}(1, n+1)$ is weakly irreducible but not irreducible. In this case, $\mathfrak{h}_1$ leaves a degenerate subspace W invariant.

In the first case, $\mathfrak{hol}_p(X, g^L)$ does not leave any lightlike line invariant. Hence, (X, g^L) is not almost decent. There is no general statement for the second case. However, if (X, g^L) is given by the last case then it is almost decent if it is time-orientable. Let us explain this fact. First, we have an $\mathfrak{h}_1$-invariant line $W \cap W^\perp$. If v is a lightlike vector in T_pX spanning the invariant line then $Hol^0(X, g^L) \subset Stab(\mathbb{R} \cdot v) \subset SO_0(T_pX)$. If we identify T_pX with $\mathbb{R}^{1,n+1}$ then it can be shown that $Stab(\mathbb{R} \cdot v) \cong (\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$. If we choose a basis $(v, e_1, \ldots, e_n, z)$ of $\mathbb{R}^{n+2}$ satisfying $g(e_i, e_j) = \delta_{ij}$, $g(v, z) = 1$ and $g(v, v) = g(z, z) = 0$ then the Lie algebra $(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n$ of $(\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$ is given by

$$\left\{ \begin{pmatrix} a & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R},\ A \in \mathfrak{so}(n),\ w \in \mathbb{R}^n \right\}.$$

Lemma 1.3. *For a Lorentzian manifold (X, g^L) whose Borel-Lichnérowicz decomposition is given by*

$$T_pX = E_0 \oplus \ldots \oplus E_\ell \quad \text{and} \quad \mathfrak{hol}_p(X, g^L) = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_\ell$$

with E_i positive definite for $i \neq 1$ and $\mathfrak{h}_1 \neq \mathfrak{so}(1, n+1)$ let $W \cap W^\perp$ be an isotropic $\mathfrak{h}_1$-invariant subspace. Then $W \cap W^\perp \subset E_1$ is invariant under the action of the full holonomy group $Hol(X, g^L)$.

Proof. The idea is to apply $Hol(X, g^L) \subset \mathrm{Norm}_{O(1,\dim X-1)}(Hol^0(X, g^L))$. For $v \in W \cap W^\perp$ we have $\mathfrak{h}_1 \cdot v \in \mathbb{R} \cdot v$. Let H_i be the connected Lie subgroup of $Hol^0(X, g^L)$ whose Lie algebra is $\mathfrak{h}_i$. For any $h \in Hol^0(X, g^L)$ we have $h(v) = \alpha_h \cdot v$ since H_i acts trivially on E_1 for $i \neq 1$. Therefore, $(g^{-1}hg)(g^{-1}v) = \alpha_h \cdot g^{-1}(v)$ for

[2] We say $V \in \Gamma(U, TX)$ is *recurrent* if $\nabla^L_\cdot V = \alpha_U(\cdot)V$ for some 1-form $\alpha_U \in \Gamma(U, T^*X)$.

$g \in \mathrm{Norm}_{O(1,\dim X-1)}(Hol^0(X,g^L))$ and $h \in Hol^0(X,g^L)$. For $0 \le i \le \ell$ let $\tilde{v}_i \in E_i$ such that $g^{-1}(v) = \tilde{v}_0 + \ldots + \tilde{v}_\ell$. Using $H_i \subset g^{-1}Hol^0(X,g^L)g$ we derive $\mathbb{R}\cdot g^{-1}(v) \ni h\cdot g^{-1}(v) = \tilde{v}_0 + \ldots + \tilde{v}_{i-1} + h\tilde{v}_i + \tilde{v}_{i+1} + \ldots + \tilde{v}_\ell$ for all $h \in H_i$. Therefore, $h\tilde{v}_i \in \mathbb{R}\cdot\tilde{v}_i$ and for $i \ge 2$ we conclude $\tilde{v}_i = 0$ since H_i acts irreducibly. Hence, $g^{-1}(v) = \tilde{v}_0 + \tilde{v}_1$.

On the other hand, we have $\mathbb{R}\cdot g^{-1}(v) \ni h\cdot g^{-1}(v) = \tilde{v}_0 + h\tilde{v}_1$ for all $h \in H_1$. Hence, $\tilde{v}_1 \in \mathbb{R}\cdot v$ since H_1 acts weakly irreducibly and reducibly on E_1. If $\tilde{v}_0 \ne 0$ we derive the contradiction $0 = \langle g^{-1}(v), g^{-1}(v)\rangle = \langle\tilde{v}_0,\tilde{v}_0\rangle + 2\langle\tilde{v}_0,\tilde{v}_1\rangle + \langle\tilde{v}_1,\tilde{v}_1\rangle = \langle\tilde{v}_0,\tilde{v}_0\rangle \ne 0$ since E_0 is definite. Therefore, $g^{-1}(v) \in \mathbb{R}\cdot v$ and $Hol(X,g^L)\cdot v \in \mathbb{R}\cdot v$. ■

We conclude that $W \cap W^\perp$ corresponds to a ∇^L-parallel lightlike subbundle $\Xi \subset TX$ of rank one. If (X,g^L) is time-orientable[3] we have a global section $V \in \Gamma(X,\Xi)$, i.e., (X,g^L,V) is almost decent.

2. A Lorentzian - Riemannian Dictionary

Let (X,g^L,V) be an almost decent spacetime and S a realization of the screen bundle. Using Cor. 1.1 we fix $Z \in \Gamma(X,\Theta)$ and define the following Riemannian metric on X.

$$g^R(A,B) := \begin{cases} 1 & \text{if } A = B = V \text{ or } A = B = Z, \\ g^L(A,B) & \text{if } A, B \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Given a choice for S and V we say g^R is the (V,S)-metric associated to g^L. We have $T\mathcal{X}^\perp = \Xi \oplus S$ and $\Theta = (T\mathcal{X}^\perp)^{\perp_{g^R}}$ where $(T\mathcal{X}^\perp)^{\perp_{g^R}}$ is the normal bundle of $T\mathcal{X}^\perp \subset TX$ w.r.t. g^R.

Definition 2.1. Let $(X,\mathcal{F})$ be a foliated manifold and $\Gamma(U,T\mathcal{F})$ the vector fields on $U \subset X$ tangent to $\mathcal{F}$.

(1) A Riemannian metric g^R on X is bundle-like w.r.t. $\mathcal{F}$ if $(L_V g)(Y_1,Y_2) = 0$ for any open subset $U \subset X$ and all $V \in \Gamma(U,T\mathcal{F})$, $Y_\cdot \in \Gamma(U,T\mathcal{F}^{\perp_{g^R}})$.
(2) We say $(X,\mathcal{F})$ is transversely parallelizable if there exists a global frame $(\bar{Y}_1,\ldots,\bar{Y}_{\mathrm{codim}\,\mathcal{F}})$ for $TX/_{T\mathcal{F}}$ and global sections $Y_i \in \Gamma(X,TX)$ such that $[Y_i,T\mathcal{F}] \in T\mathcal{F}$ and $\bar{Y}_i = pr_{TX/_{T\mathcal{F}}}(Y_i)$ for all $1 \le i \le \mathrm{codim}\,\mathcal{F}$. ■

Lemma 2.2. *Let (X,g^L,V) be an almost decent spacetime. For any realization of the screen bundle S the following are equivalent.*

(1) *The (V,S)-metric g^R is bundle-like w.r.t. $\mathcal{X}^\perp$ and $(X,\mathcal{X}^\perp)$ is transversely parallelizable,*
(2) *the 1-form $g^L(V,\cdot)$ defining $\mathcal{X}^\perp$ is closed and*
(3) *(X,g^L) is decent, i.e., $\alpha|_{\Xi^\perp} = 0$.*

Proof. Suppose $\alpha|_{\Xi^\perp} = 0$. Let $V \in \Gamma(X,\Xi)$ and fix $Z \in \Gamma(X,\Theta)$. We have to show $(L_W g^R)(Z,Z) = 0$ for all $W \in \Gamma(U,\Xi^\perp)$. Using $g^R(\cdot,Z) = g^L(\cdot,V)$ we derive $g^R(\nabla^L_W Z, Z) = g^L(\nabla^L_W Z, V) = -g^L(Z,\nabla^L_W V) = -\alpha(W)$ and

$$\begin{aligned}(L_W g^R)(Z,Z) &= \underbrace{W(g^R(Z,Z))}_{=0} - 2g^R([W,Z],Z) = 2g^R(\underbrace{\nabla^L_Z W}_{\in\Xi^\perp}, Z) - 2g^R(\nabla^L_W Z, Z) \\ &= 2\alpha(W).\end{aligned}$$

[3] If (X,g) is not time-orientable we may consider its 2-fold time-orientation cover. The global nowhere vanishing section $V \in \Gamma(X,\Xi)$ is recurrent but not necessarily parallel even if $\mathfrak{h}_1$ annihilates a vector. In fact, we derive a class in $H^1(X,\mathbb{R})$ induced by $\pi_1(X) \twoheadrightarrow Hol_p(\nabla^L)/Hol^0_p(\nabla^L) \to \mathbb{R}$ where the last morphism is induced by $pr_{\Xi_p} \circ Hol_p(\nabla^L)|_{\Xi_p}$.

Thus, g^R is bundle-like w.r.t. $\mathcal{X}^\perp$. Moreover, $pr_Z([W,Z]) := g^R([W,Z],Z)Z = -\alpha(W)Z$ and Z is globally defined, i.e., the foliation $\mathcal{X}^\perp$ is transversely parallelizable. For the last statement we compute

$$\begin{aligned} d(g^L(V,\cdot))(W,Z) &= g^L(\nabla^L_W V, Z) - g^L(\nabla^L_Z V, W) \\ &= \alpha(W) g^L(V,Z) - \alpha(Z) g^L(V,W) = \alpha(W). \end{aligned}$$

For the converse we follow these equations backwards. ■

Lemma 2.3. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\Xi^\perp$. Let S be any realization of the screen bundle.*

1. *The restriction $g^R|_{\mathcal{L}^\perp}$ of the (V,S)-metric is bundle-like w.r.t. the foliation $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$.*[4]
2. *The (V,S)-metric is bundle-like w.r.t. the foliation $(X,\mathcal{X})$ if and only if $\alpha(V) = 0$*[5] *and $[V,Z] \in \Gamma(X,\Xi)$.*

Proof. Since $g^R|_{S\times S} = g^L|_{S\times S}$ we have for any $Y_1, Y_2 \in \Gamma(U,S)$

$$\begin{aligned} (L_V g^R)(Y_1,Y_2) &= V(g^L(Y_1,Y_2)) - g^L([V,Y_1],Y_2) - g^L([V,Y_2],Y_1) \\ &= g^L(\nabla^L_V Y_1, Y_2) + g^L(Y_1, \nabla^L_V Y_2) - g^L([V,Y_1],Y_2) - g^L([V,Y_2],Y_1) \\ &= g^L(\nabla^L_{Y_1} V, Y_2) + g^L(Y_1, \nabla^L_{Y_2} V) = 0 \end{aligned}$$

For the second statement we need to show $(L_V g^R)(Y_1,Y_2) = 0$ for any $Y_1, Y_2 \in \Gamma(U, S\oplus\Theta)$. If $Y_1 = Z$ and $Y_2 \in \Gamma(U,S)$ we derive

$$\begin{aligned} (L_V g^R)(Z,Y_2) &= V(g^R(Z,Y_2)) - g^R([V,Z],Y_2) - g^R([V,Y_2],Z) \\ &= -g^L([V,Z],Y_2) - g^L(\underbrace{[V,Y_2]}_{\in\Xi^\perp}, V) = -g^L([V,Z],Y_2). \end{aligned}$$

For $Y_1 = Y_2 = Z$ we have $(L_V g^R)(Z,Z) = V(g^R(Z,Z)) - 2g^R([V,Z],Z) = 2\alpha(V)$. Since $g^L(\nabla^L_V Z, V) = -\alpha(V)$ we conclude $[V,Z] \in \Gamma(X,\Xi)$ if $\alpha(V) = 0$ and $[V,Z] \in \Gamma(X, \Xi\oplus\Theta)$. ■

For a foliated manifold $(X,\mathcal{F})$ let $X/_{\mathcal{F}}$ be its set of leaves and

$$\pi : X \to X/_{\mathcal{F}}, \qquad p \mapsto \text{(leaf through } p).$$

For Riemannian foliations this map has been studied in [Her60], [Rei61], [Esc82] and [Mol88]. Given the results in [Con74] and Lemma 2.2 we have

Corollary 2.4 (Conlon [Con74])**.** *For a decent spacetime (X, g^L, V) all leaves of $(X,\mathcal{X}^\perp)$ have trivial leaf-holonomy. Suppose there is a realization of the screen bundle such that Z is complete and let $\mathcal{L}^\perp$ be a leaf of $\mathcal{X}^\perp$.*

1. *If there is no leaf of $\mathcal{X}^\perp$ which is closed in X then each leaf is dense in X.*
2. *We have $\tilde{X} = \tilde{\mathcal{L}^\perp} \times \mathbb{R}$ where $\tilde{X}$, $\tilde{\mathcal{L}^\perp}$ denote the universal covers of X, $\mathcal{L}^\perp$.*
3. *If there is a closed leaf then $X \to X/_{\mathcal{X}^\perp}$ is a smooth fiber bundle and $X/_{\mathcal{X}^\perp} \in \{\mathbb{R}, S^1\}$.*
4. *The inclusion $\mathcal{L}^\perp \to X$ induces a monomorphism $\pi_1(\mathcal{L}^\perp) \to \pi_1(X)$ onto a normal subgroup. If X is compact then $\pi_1(X)/_{\pi_1(\mathcal{L}^\perp)} = \mathbb{Z}^r$ for some $r \geq 1$ and $r = 1$ if and only if $\mathcal{L}^\perp$ is closed in X.* ■

A spacetime (X, g^L) is said to be distinguishing at $p \in X$ if for any neighborhood $U \ni p$ there is a neighborhood $V \subset U$ such that $p \in V$ and any (piecewise smooth) causal curve $\gamma : [a,b] \to X$ with $\gamma(a) = p$ and $\gamma(b) \in V$ is contained in V. We say

[4] This fact seems to be well known and the first reference I could find is [Zeg99].

[5] The integral curves of V are g^L-geodesics if and only if $\alpha(V) = 0$ since $\nabla^L_{\cdot} V = \alpha(\cdot)V$.

(X, g^L) is a distinguishing spacetime if it is distinguishing for all $p \in X$. On the causality ladder (cf. [MS08]) we have

$$\text{strongly causal} \Rightarrow \text{distinguishing} \Rightarrow \text{causal}.$$

Proposition 2.5. *Let (X, g^L, V) be an almost decent spacetime.*

(1) *If (X, g^L) is causal then the leaves of the foliated manifolds $(X, \mathcal{X})$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ have trivial leaf holonomy. Moreover, X is not compact.*
(2) *If (X, g^L) is distinguishing at $p \in X$ then the leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ through p is not compact.*
(3) *If (X, g^L) is distinguishing then each leaf of $\mathcal{X}$ is a closed subset in X and each leaf of $\mathcal{X}|_{\mathcal{L}^\perp}$ is a closed subset in $\mathcal{L}^\perp$.*

Proof. Any curve in a leaf $\mathcal{L}$ of $\mathcal{X}$ is lightlike, i.e., $\pi_1(\mathcal{L}) = 0$ since (X, g^L) is causal.

Suppose $\mathcal{L}^\perp$ is compact. There is a bundle-like Riemannian metric on the compact foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$. Consider the leaf $\mathcal{L} \subset \mathcal{L}^\perp$ of $\mathcal{X}|_{\mathcal{L}^\perp}$ through p. If $\mathcal{L}$ is closed in $\mathcal{L}^\perp$ then it is compact, i.e., we have a closed lightlike curve through p. In this case (X, g^L) would not be causal at p. On the other hand, if $\mathcal{L}$ is not closed in $\mathcal{L}^\perp$ then $\bar{\mathcal{L}} \subset \mathcal{L}^\perp$ is diffeomorphic to a torus in $\mathcal{L}^\perp$ by [Car84] and (X, g^L) is not distinguishing at p.

Let $\mathcal{L} \subset \mathcal{L}^\perp$ be a leaf of $\mathcal{X}$. Suppose we have $q \in \bar{\mathcal{L}} \setminus \mathcal{L}$ where the closure is taken w.r.t. X. For any X-open neighborhood $U \ni q$ we can find $p_U \in \mathcal{L} \cap U$. In particular, we may choose U to be a coordinate neighborhood ball such that $\bar{U} \subset \tilde{U}$ where $\tilde{U}$ is a Walker coordinate neighborhood, i.e., $g^L = 2dxdz + u_\alpha dy^\alpha dz + f dz^2 + g_{\alpha\beta} dy^\alpha dy^\beta$ in $\tilde{U}$ and $V \in \operatorname{span}\{\partial_x\}$. In these coordinates we have $p_U = (x_0, y_0^1 \ldots, y_0^n, z_0)$ and a curve segment $\gamma : [0, b] \to U$ with $t \mapsto (t + x_0, y_0^1, \ldots, y_0^n, z_0)$. Thus, $\dot{\gamma} \in \Xi$ implies $\gamma([0, b]) \subset \mathcal{L}$ and since $\bar{U} \subset \tilde{U}$ we may assume $\gamma(b) \notin U$. Finally, we can find $\tilde{p}_U \in \mathcal{L} \cap U$ such that $\tilde{p}_U \notin \{(\cdot, y_0^1, \ldots, y_0^n, z_0)\}$ and since $\mathcal{L}$ is connected there is a curve $\tilde{\gamma}$ in $\mathcal{L}$ connecting $\gamma(b)$ and $\tilde{p}_U$. Therefore, we have a (piecewise smooth) lightlike curve from p_U to $\tilde{p}_U$ which leaves U and (X, g^L) is not distinguishing at q. Hence, $\mathcal{L}$ is closed in X and being the preimage of a closed set under $\mathcal{L}^\perp \to X$ it is closed in $\mathcal{L}^\perp$. ∎

Consider a leaf $\mathcal{L}^\perp$ in a distinguishing almost decent spacetime. By Prop. 2.5 all leaves of the foliation $\mathcal{X}|_{\mathcal{L}^\perp}$ are closed with vanishing fundamental group. Hence, $\mathcal{L}^\perp \to \mathcal{L}^\perp/\chi$ is a submersion if $\mathcal{L}^\perp/\chi$ is Hausdorff [Sha97]. Since $g^R|_{\mathcal{L}^\perp}$ is bundle-like a sufficient condition is that all segments γ of horizontal geodesics, i.e., $\dot{\gamma}(0) \in S|_{\mathcal{L}^\perp}$, can be indefinitely extended (cf. [Her60]). However, for $Y_. \in \Gamma(U, S)$ the Koszul formula implies $g^R(\nabla^R_{Y_1} Y_2, Y_3) = g^L(\nabla^L_{Y_1} Y_2, Y_3)$. Hence, a curve γ tangent to S is a horizontal geodesic if $\nabla^S_{\dot{\gamma}} \dot{\gamma} = 0$.

Example 2.6. Let $(M, \tilde{g})$ be a simply connected compact Riemannian manifold and $f \in C^\infty(M)$. For $\varepsilon > 0$ and $L \in \{\mathbb{R}, S^1\}$ define $X := S^1 \times L \times M$ and

$$g^L_\varepsilon := 2dxdz + \varepsilon f dz^2 + \tilde{g}$$

where dx and dz are the standard coordinate 1-forms on $S^1 \times L$. If $f \in C^\infty(M)$ is suitable then (X, g^L_ε) is weakly irreducible where ∂_x is $\nabla^{g^L_\varepsilon}$-parallel. Moreover, the leaves of $(X, \mathcal{X}^\perp)$ are compact and the universal cover of (X, g^L_ε) is globally hyperbolic if ε is sufficiently small.

Proof. Each leaf of $\mathcal{X}^\perp$ is diffeomorphic to $S^1 \times M$ and the universal cover of X is given by $\mathbb{R}^2 \times M$. The pullback of g^L_ε to $\mathbb{R}^2 \times M$ is of the form $2dxdz + \varepsilon f dz^2 + g$ where x and z are the standard coordinates on $\mathbb{R}^2$. Bazaikin has shown in [Baz09, Thm. 2] that this metric is globally hyperbolic if ε is sufficiently small. ∎

For a Riemannian foliation $(X,\mathcal{F})$ with a bundle-like metric g^R the *transverse Levi-Civita connection* ∇^T on $(T\mathcal{F})^\perp$ is given by

$$\nabla^T_X Y = \begin{cases} \pi_{(T\mathcal{F})^\perp}(\nabla^{g^R}_X Y) & X \in (T\mathcal{F})^\perp, \\ \pi_{(T\mathcal{F})^\perp}([X,Y]) & X \in T\mathcal{F}, \end{cases}$$

where $Y \in \Gamma(U,(T\mathcal{F})^\perp)$. Consider the foliation $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ with the (V,S)-metric $g^R|_{\mathcal{L}^\perp}$. Any local section $\tilde{V} \in \Gamma(U,\Xi)$ is given by $\tilde{V} = fV$. Hence,

$$\nabla^T_{\tilde{V}} Y = \pi_{(T\mathcal{F})^\perp}(\nabla^L_{\tilde{V}} Y) - \pi_{(T\mathcal{F})^\perp}(\underbrace{\nabla^L_Y fV}_{\in\Xi}) = \pi_{(T\mathcal{F})^\perp}(\nabla^L_{\tilde{V}} Y) = \nabla^S_{\tilde{V}} Y$$

and using $g^R(\nabla^R_{Y_1} Y_2, Y_3) = g^L(\nabla^L_{Y_1} Y_2, Y_3)$ we conclude

Corollary 2.7. *Let (X,g^L,V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. For any realization S of the screen bundle the transverse Levi-Civita connection coincides with $\nabla^S|_{\mathcal{L}^\perp}$.* ■

Definition 2.8. Let (X,g^L,V) be an almost decent spacetime. If S is a realization of the screen bundle we say (X,g^L,V,S) is

- almost horizontal if $\alpha(Y) = g^L(Z,\nabla^L_V Y)$ or equivalently $[V,Y] \in S$ for any local section $Y \in \Gamma(U,S)$,
- horizontal if it is almost horizontal and decent. ■

Hence, $\nabla^L_V Y \in \Gamma(U,S)$ for any section $Y \in \Gamma(U,S)$ if (X,g^L,V,S) is horizontal. In particular, $d(g^L(Z,\cdot))(V,\cdot)|_{\Xi^\perp} = -g^L(Z,[V,\cdot])|_{\Xi^\perp} = 0$ if and only if (X,g^L,V,S) is almost horizontal.

Lemma 2.9. *Let (X,g^L,V) be an almost decent spacetime. If S is a realization of the screen bundle then*

(1) *(X,g^L,V,S) is almost horizontal if and only if for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ the restriction of $g^R|_{\mathcal{L}^\perp}$ of the (V,S)-metric defines the structure of an isometric Riemannian flow on $(\mathcal{L}^\perp,\mathcal{X}|_{\mathcal{L}^\perp})$, i.e., $L_V g^R(W_1,W_2) = 0$ for all $W_1,W_2 \in \Xi^\perp$ and V is a $g^R|_{\mathcal{L}^\perp}$-Killing vector field of constant length.*
(2) *The (V,S)-metric is bundle-like w.r.t. the foliation $(X,\mathcal{X})$ and $\alpha|_S = 0$ if and only if (X,g^L,V,S) is horizontal.*
(3) *The (V,S)-metric g^R defines the structure of an isometric Riemannian flow on $(X,\mathcal{X})$ and $\alpha|_S = 0$ if and only if (X,g^L,V,S) is horizontal and $\alpha = 0$.*

Proof. Lemma 2.3 implies $L_V g^R(W_1,W_2) = 0$ for all $W_1,W_2 \in S$ and the first equivalence follows from

$$L_V g^R(V,W_2) = V(g^R(V,W_2)) - g^R([V,V],W_2) - g^R([V,W_2],V) = -g^L([V,W_2],Z).$$

If (X,g^L,V,S) is horizontal we have a local orthonormal frame $(Y_1,\ldots,Y_{\dim S})$ for S such that $[V,Y_i] \in S$. Thus, $g^L(\nabla^L_V Z, Y_i) = -g^L(Z,\nabla^L_V Y_i) = 0$ and $pr_S([V,Z]) = pr_S(\nabla^L_V Z)$ imply $[V,Z] \in \Gamma(X,\Xi)$. This implies the second equivalence by Lemma 2.3. For the last statement we consider $L_V g^R(V,Z) = -g^R([V,Z],V) = \alpha(Z)$. ■

Proposition 2.10. *Let (X,g^L,V,S) be a horizontal spacetime. If $f \in C^\infty(X)$ is $(X,\mathcal{X})$-basic, i.e., $V(f) = 0$, then (X,g^f) is a horizontal spacetime where the transverse conformal change g^f of g^L by f is defined by*

$$g^f := \begin{cases} g^f|_{S\times S} = e^f g^L|_{S\times S}, \\ g^f(V,V) = g^f(Z,Z) = g^f(V,S) = g^f(Z,S) = 0, \\ g^f(V,Z) = 1. \end{cases}$$

Proof. First, we show that $\nabla^f_\cdot V = \nabla^L_\cdot V$. Let $(V, Y_1, \dots, Y_{\dim S}, Z)$ be a local frame of (X, g^L) where $(Y_\alpha)_\alpha$ is a g^L-orthonormal frame for S. The Koszul formula and $\alpha|_{\Xi^\perp} = 0$ imply for $U_1, U_2 \in \{V, Y_\cdot, Z\}$

$$2g^f(\nabla^f_{U_1} V, U_2) = V g^f(U_1, U_2) + g^f([U_1, V], U_2) - g^f([V, U_2], U_1).$$

If $U_1 = U_2 = Z$ we derive $2g^f(\nabla^f_Z V, Z) = 2g^L(\nabla^L_Z V, Z) = 2\alpha(Z)$. If $U_1, U_2 \in \Xi^\perp$ we have $[V, U_\cdot] \in \Xi^\perp$. Hence,

$$\begin{aligned} 2g^f(\nabla^f_{U_1} V, U_2) &= V(e^f) g^L(U_1, U_2) \\ &\qquad + e^f(V g^L(U_1, U_2) + g^L([U_1, V], U_2) - g^L([V, U_2], U_1)) \\ &= V(e^f) g^L(U_1, U_2) = 0 \end{aligned}$$

since f is $(X, \mathcal{X})$-basic. If $U_1 = Z$ and $U_2 \in S$ we conclude $2g^f(\nabla^f_{U_1} V, U_2) = e^f g^L([Z, V], U_2) - g^L([V, U_2], Z) = (e^f - 1) g^L(Z, \nabla^L_V U_2) = 0$ since (X, g^L) is horizontal. The case $U_1 \in S$ and $U_2 = Z$ is similar. On the other hand, $U_1 = V$ and $U_2 = Z$ implies $2g^f(\nabla^f_{U_1} V, U_2) = -g^L([V, Z], V) = \alpha(V) = 0$. Finally,

$$\begin{aligned} 2g^f(\nabla^f_V Y_\cdot, Z) &= g^f(\underbrace{[V, Y_\cdot]}_{\in S}, Z) - g^f([V, Z], Y_\cdot) - g^f([Y_\cdot, Z], V) \\ &= e^f g^L(Z, \underbrace{\nabla^L_V Y_\cdot}_{\in S}) + g^L(Z, \nabla^L_{Y_\cdot} V) = 0. \end{aligned}$$

Hence, (X, g^f) is horizontal. ∎

If (X, g^L) is a Walker coordinate neighborhood of the form $g^L = 2dxdz + u_\alpha dy^\alpha dz + hdz^2 + g_{\alpha\beta} dy^\alpha dy^\beta$ and we choose $V := \partial_x$ and $Z := \partial_z - \frac{1}{2} h \partial_x$ then the transverse conformal change is given by $g^f = 2dxdz + u_\alpha dy^\alpha dz + hdz^2 + e^f g_{\alpha\beta} dy^\alpha dy^\beta$.

If (X, g^L, V, S) is horizontal then $[V, Z] \in \Gamma(X, \Xi)$, i.e., V and Z induce a 2-dimensional foliation on X. The (V, S)-metric g^R is bundle-like w.r.t. this foliation if $(L_Z g^R)|_{S \times S} = 0$. If (X, g^L) is a Walker coordinate neighborhood as above this condition corresponds to $\partial_z g_{\alpha\beta} = 0$.

Corollary 2.11. *Let (X, g^L, V, S) be an almost horizontal spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. If all leaves of $\mathcal{X}|_{\mathcal{L}^\perp}$ are compact then the projection $\mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ is a principal S^1-orbibundle over $\mathcal{L}^\perp/\mathcal{X}|_{\mathcal{L}^\perp}$ for which $S|_{\mathcal{L}^\perp}$ defines a connection whose connection 1-form is $g^L(Z, \cdot)$.*

Proof. Since (X, g^L, V, S) is almost horizontal $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ is an isometric flow and the statement follows from [Mol88, Prop. 3.7] and a theorem of Wadsley. See [BG08, Thm. 6.3.8] or [SS09, Prop. 3] for the details. ∎

The following examples of horizontal spacetimes show that the leaves of $\mathcal{X}$ and $\mathcal{X}^\perp$ are not necessarily closed.

Example 2.12. Let $(M := S^1 \times S^1, g)$ be the flat torus and $a \in \mathbb{R} \setminus \mathbb{Q}$. Write ∂_x, ∂_y for the standard coordinate vector fields on M and define $\eta := g(\partial_x - \frac{1}{a}\partial_y, \cdot)$. The trivial S^1-bundle $S^1 \times T^2$ admits a weakly irreducible horizontal Lorentzian metric g^L such that the leaves of $\mathcal{X}$ are the fibers of the bundle. Moreover, all leaves of $\mathcal{X}^\perp$ are dense in $S^1 \times T^2$.

Proof. The construction of the metric is a special case of [Lär08, Prop. 3.1]. There the foliation $\mathcal{X}^\perp$ is defined by $\pi^*\eta$. If $\eta := g(\partial_x - \frac{1}{a}\partial_y, \cdot)$ then $\mathrm{Ker}\eta = \mathrm{span}\{\partial_x + a\partial_y\}$. Hence, the leaves of $\pi^*\eta$ are dense in $S^1 \times T^2$. ∎

Example 2.13. Let $X := T^2 \times S^1$ where $T^2 := S^1 \times S^1$ and $a \in \mathbb{R} \setminus \mathbb{Q}$. Write ∂_x, ∂_y for the standard coordinate vector fields of T^2 and ∂_z for the last standard coordinate vector field in $T^2 \times S^1$. For $f \in C^\infty(T^2)$ define g_f^L by

$$\begin{aligned} g_f^L(\partial_x + a\partial_y, \partial_x + a\partial_y) &= g_f^L(\partial_x + a\partial_y, \partial_y) = g_f^L(\partial_y, \partial_z) = 0, \\ g_f^L(\partial_x + a\partial_y, \partial_z) &= g_f^L(\partial_y, \partial_y) = 1 \text{ and } g_f^L(\partial_z, \partial_z) = f. \end{aligned}$$

Then we have $\nabla^{g_f^L}_{\cdot}(\partial_x + a\partial_y) = \alpha(\cdot)(\partial_x + a\partial_y)$ such that $\alpha|_{\Xi^\perp} = 0$ and $Hol(X, g_f^L) = \mathbb{R} \ltimes \mathbb{R}$ for a suitable choice of f. In particular, $\mathcal{L}^\perp = T^2$ for any leaf of $\mathcal{X}^\perp$ and all leaves of $\mathcal{X}$ are dense in T^2. Finally, $(X, g_f^L, \partial_x + a\partial_y, \operatorname{span}\{\partial_y\})$ is horizontal.

Proof. Define $V := \partial_x + a\partial_y$. Then $[V, \partial_y] = [V, \partial_z] = [\partial_y, \partial_z] = 0$, i.e., locally we have coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ such that $V = \partial_{\tilde{x}}$, $\partial_y = \partial_{\tilde{y}}$ and $\partial_z = \partial_{\tilde{z}}$. In these local coordinates the Lorentzian metric is given by

$$g_f^L = 2d\tilde{x}d\tilde{z} + f d\tilde{z}^2 + d\tilde{y}^2.$$

Since $V = \partial_{\tilde{x}}$ we conclude that $\nabla^L_{\cdot} V = \alpha(\cdot)V$ and $\alpha|_{\Xi^\perp} = 0$. If the restriction of g^L to the coordinate neighborhood U is weakly irreducible so is (X, g_f^L). Using a suitable choice of $f \in C^\infty(T^2)$ we derive a weakly irreducible neighborhood (U, g_f^L) and $\frac{\partial^2 f}{\partial \tilde{x}^2} \neq 0$.

The maximal integral curves of V are dense in T^2 since $a \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, $\Xi^\perp = \operatorname{span}\{V, \partial_y\}$, i.e., $\mathcal{L}^\perp = T^2$. The global vector field ∂_y defines a non-canonical realization S of the screen bundle. Hence, S admits a global nowhere vanishing section which is ∇^S-parallel and we conclude $Hol(X, g_f^L) = \mathbb{R} \ltimes \mathbb{R}$. Finally, $\partial_y = \partial_{\tilde{y}}$ and the local coordinate structure imply $\nabla^L_V \partial_y = 0$, i.e., (X, g_f^L) is horizontal. ∎

Example 2.6 is in fact horizontal if $V := \partial_x$ and $S := TM$ and another class of globally hyperbolic decent spacetimes was constructed in [BM08]. Using the notation of [BM08] we derive horizontal spacetimes if $S := TF$. Finally, Tom Krantz constructed another class of weakly irreducible spacetimes which are almost horizontal by [Kra10, Prop. 4].

3. Ricci Comparison for Decent Spacetimes

Let $(X, \mathcal{F})$ be a foliated manifold. A differential r-form ω on X is X-basic or basic if $V \lrcorner \omega = 0$ and $L_V \omega = 0$. We derive a sheaf of germs of basic r-forms and write $\Lambda^r_B \mathcal{F}$ for its space of global sections. By definition, if ω is basic so is $d\omega$. Hence, we have the *basic cohomology ring* $H^*_B(X, \mathcal{F})$ of $(X, \mathcal{F})$. If X is connected we have $H^0_B(X, \mathcal{F}) = \mathbb{R}$ and a group monomorphism $H^1_B(X, \mathcal{F}) \hookrightarrow H^1(X, \mathbb{R})$ induced by $\Lambda^1_B \mathcal{F} \hookrightarrow \Lambda^1 TX$ (cf. [BG08][Prop. 2.4.1]).

Let (X, g^L, V) be a decent spacetime and consider the (V, S)-metric g^R for some realization S of the screen bundle. If X is compact we have $b_1(X) \geq 1$ by Cor.2.4 and if $\mathcal{X}^\perp$ admits a compact leaf the projection onto the space of leaves is a fiber bundle $X \to S^1$ whose fibers are the leaves of $\mathcal{X}^\perp$. Hence, X is a mapping torus, i.e., if $\mathcal{L}^\perp$ is a leaf of $\mathcal{X}^\perp$ there is a diffeomorphism F of $\mathcal{L}^\perp$ such that $X = \mathcal{L}^\perp \times [0,1]/_\sim$ where $(p, 0) \sim (F(p), 1)$. Using Cor. 2.4 we have $b_1(X) = b_1(\mathcal{L}^\perp) + 1$. For the higher Betti numbers a Mayer-Vietoris argument yields the following exact sequence in singular homology

$$\longrightarrow H_i(\mathcal{L}^\perp) \xrightarrow{Id - F^i_*} H_i(\mathcal{L}^\perp) \xrightarrow{\iota_*} H_i(X) \longrightarrow H_{i-1}(\mathcal{L}^\perp) \xrightarrow{Id - F^{i-1}_*} H_{i-1}(\mathcal{L}^\perp) \longrightarrow$$

where F^i_* is the morphism induced by F and $\iota : \mathcal{L}^\perp \hookrightarrow X$ is the inclusion. On the other hand, if X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then the natural projection induces a fiber bundle map $X \to \mathbb{R}$ [Sha97], i.e., $X \cong \mathcal{L}^\perp \times \mathbb{R}$ and $b_i(X) = b_i(\mathcal{L}^\perp)$.

Consider an arbitrary almost decent spacetime (X, g^L, V) and suppose $\mathcal{X}^\perp$ admits a compact leaf $\mathcal{L}^\perp$. If $\tilde{g}^R$ is a bundle-like Riemannian metric on the compact foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ and $E^\perp$ is the $\tilde{g}^R$-orthogonal complement of V we define the mean curvature 1-form by $\kappa_{\tilde{g}^R} := \tilde{g}^R(pr_{E^\perp}(\nabla^{\tilde{g}^R}_{\frac{V}{\|V\|_{\tilde{g}^R}}} \frac{V}{\|V\|_{\tilde{g}^R}}), \cdot)$. Since $\mathcal{L}^\perp$ is compact [Dom98] and [Mas00] imply the existence of a bundle-like Riemannian metric $\tilde{g}^B$ on $\mathcal{L}^\perp$ such that $\kappa_{\tilde{g}^B}$ is basic and harmonic w.r.t. the basic Laplacian. In this case, the Euler form $\mathbf{e}$ of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, \tilde{g}^B)$ is defined using Rummler's formula

$$d(\tilde{g}^B(\frac{V}{\|V\|}, \cdot)) = \tilde{g}^B(\frac{V}{\|V\|}, \cdot) \wedge \kappa_{\tilde{g}^B} + \mathbf{e}.$$

In [RP01] Royo Prieto proved the existence of a Gysin sequence for $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^B)$ relating the basic cohomology of $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ to the cohomology of $\mathcal{L}^\perp$ by[6]

$$\cdots \to H^i_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to H^i(\mathcal{L}^\perp, \mathbb{R}) \to H^{i-1}_{d-\kappa_{\tilde{g}^B}}(\mathcal{X}|_{\mathcal{L}^\perp}) \overset{[\cdot \wedge \mathbf{e}]}{\to} H^{i+1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to \cdots$$

Here, we write $H^*_{d-\kappa_{\tilde{g}^B}}(\mathcal{X}|_{\mathcal{L}^\perp})$ for the *dual basic cohomology* which can be defined in the following way. If $(X, \mathcal{F}, \tilde{g}^B)$ is a Riemannian flow whose mean curvature $\kappa_{\tilde{g}^B}$ 1-form is basic and harmonic w.r.t. the basic Laplacian then $H^*_{d-\kappa_{\tilde{g}^B}}(X, \mathcal{F})$ is the cohomology of the complex $(\Lambda^*_B\mathcal{F}, d - \kappa_{\tilde{g}^B} \wedge \cdot)$. It can be shown that $H^i_B(X, \mathcal{F}) \cong H^{\dim \mathcal{L}^\perp - i}_{d-\kappa_{\tilde{g}^B}}(X, \mathcal{F})$ for all $i \geq 0$ [HR10, Sec. 1.5].

For a Riemannian flow $(X, \mathcal{F}, \tilde{g}^B)$ whose mean curvature 1-form $\kappa_{\tilde{g}^B}$ is basic and harmonic consider the twisted differential $d_\kappa := d - \frac{1}{2}\kappa_{\tilde{g}^B} \wedge$. The *twisted basic cohomology* $H^*_{tw}(X, \mathcal{F})$ is defined as the cohomology of the complex $(\Lambda^*_B\mathcal{F}, d_\kappa)$ and if δ_κ denotes the formal L^2-adjoint of d_κ on $\Lambda^*_B\mathcal{F}$ the twisted basic Laplacian is defined by $\Delta_\kappa := d_\kappa\delta_\kappa + \delta_\kappa d_\kappa$. In [HR10] Habib and Richardson proved a Hodge decomposition theorem for Δ_κ and the following Weitzenböck formula for any basic form $\varphi \in \Lambda^*_B\mathcal{F}$

$$\Delta_\kappa\varphi = \nabla^{T^*}\nabla^T\varphi + \sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i, e_j)\varphi + \frac{1}{4}|\kappa_{\tilde{g}^B}|^2\varphi.$$

Here we write ∇^{T^*} for the formal L^2-adjoint of the transverse Levi-Civita connection on basic forms and $R^T(e_i, e_j) := [\nabla^T_{e_i}, \nabla^T_{e_j}] - \nabla^T_{[e_i, e_j]}$ where $(e_1, \ldots, e_{\dim X - 1})$ is a transverse orthonormal frame. For a basic 1-form φ Habib and Richardson proved

$$\langle\Delta_\kappa\varphi, \varphi\rangle = \langle\nabla^{T^*}\nabla^T\varphi, \varphi\rangle + Ric^T(\varphi^\sharp, \varphi^\sharp) + \frac{1}{4}|\kappa_{\tilde{g}^B}|^2|\varphi|^2,$$

where Ric^T is the Ricci curvature of ∇^T.

If (X, g^L, V) is an almost decent spacetime and S a realization of the screen bundle then Cor. 2.7 implies $\nabla^S|_{\mathcal{L}^\perp} = \nabla^T$ where ∇^T is transverse Levi-Civita connection of the Riemannian flow $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$. If $(Y_1, \ldots, Y_{\dim S})$ is a local

[6]We have seen in Lemma 2.9 that $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ is an isometric Riemannian flow if g^R is the (V, S)-metric of an almost horizontal spacetime (X, g^L, V, S). Thus, $\kappa_{g^R|_{\mathcal{L}^\perp}} = 0$ and the Gysin sequence for $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$ is given by

$$\cdots \to H^i_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to H^i(\mathcal{L}^\perp, \mathbb{R}) \to H^{i-1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \overset{\delta}{\to} H^{i+1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to \cdots$$

where $\delta = [dg^L(Z, \cdot) \wedge \cdot]$ (cf. [BG08, Thm. 7.2.1]). In particular, the Euler class is given by $[dg^L(Z, \cdot)] \in H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$.

orthonormal frame of S we write $E_\pm := \frac{1}{\sqrt{2}}(V \pm Z)$ and conclude

$$\begin{aligned}
Ric^L(Y_\alpha, Y_\beta) &= -g^L(R^L(Y_\alpha, E_-)E_-, Y_\beta)) + \sum_{k=1}^{\dim S} g^L(R^L(Y_\alpha, Y_k)Y_k, Y_\beta) \\
&\quad + g^L(R^L(Y_\alpha, E_+)E_+, Y_\beta)) \\
&= \underbrace{g^L(R^L(Y_\alpha, V)Z, Y_\beta)}_{=-g^L(R^L(Z,Y_\beta)V,Y_\alpha)} + g^L(\underbrace{R^L(Y_\alpha, Z)V}_{\in \Xi}, Y_\beta) \\
&\quad + \sum_{k=1}^{\dim S} g^L(R^L(Y_\alpha, Y_k)Y_k, Y_\beta) \\
&= \sum_{k=1}^{\dim S} g^R(R^S(Y_\alpha, Y_k)Y_k, Y_\beta) = Ric^T(Y_\alpha, Y_\beta)
\end{aligned}$$

for the Ricci curvature Ric^L of (X, g^L). Note that $Ric^L(V, \cdot)|_{\Xi^\perp} = 0$.

Proposition 3.1. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. If $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$ then $b_1(\mathcal{L}^\perp) \leq \dim \mathcal{L}^\perp$. If additionally $Ric^L_q(W, W) > 0$ for some $q \in \mathcal{L}^\perp$ and all $W \in S_q$ then $b_1(\mathcal{L}^\perp) \leq 1$.*

Proof. Suppose $Ric^L(W, W) \geq 0$ and let $\tilde{g}^B$ be a bundle-like Riemannian metric on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ having a basic and harmonic mean curvature form $\kappa_{\tilde{g}^B}$. By the Hodge theorem for the basic Laplacian [EKA90] a class $[\varphi] \in H^1_B(\mathcal{X}|_{\mathcal{L}^\perp})$ can be represented by a basic 1-form φ such that $d\varphi = \delta_B \varphi = 0$ where δ_B is the L^2-adjoint of $d|_{\Lambda^1_B \mathcal{X}|_{\mathcal{L}^\perp}}$. In this case, the Weitzenböck formula (cf. [HR10, Thm. 6.16]) has the form $0 = \int_{\mathcal{L}^\perp} |\nabla^T \varphi|^2 + \int_{\mathcal{L}^\perp} Ric^T(\varphi^\sharp, \varphi^\sharp)$. Hence, $\nabla^T \varphi = 0$, i.e., $\dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) \leq \dim \mathcal{S}$ for dimensional reasons. If $Ric^L_q(Y, Y) > 0$ at $q \in \mathcal{L}^\perp$ [HR10, Cor. 6.17] implies $H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) = 0$. Since $H^0_{d-\kappa_{\tilde{g}^B}}(\mathcal{X}|_{\mathcal{L}^\perp}) = H_B^{\dim \mathcal{L}^\perp - 1}(\mathcal{X}|_{\mathcal{L}^\perp}) \in \{\mathbb{R}, 0\}$ and

$$0 \to H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) \to H^1(\mathcal{L}^\perp, \mathbb{R}) \to H^0_{d-\kappa_{\tilde{g}^B}}(\mathcal{X}|_{\mathcal{L}^\perp}) \overset{[\cdot \wedge \mathbf{e}]}{\to} H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$$

we conclude $b_1(\mathcal{L}^\perp) \leq \dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$. ∎

Corollary 3.2. *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose $Ric^L(W, W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

1. *If X is compact and $\mathcal{X}^\perp$ admits a compact leaf then $1 \leq b_1(X) \leq \dim X$.*
2. *If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $0 \leq b_1(X) \leq \dim X - 1$.*

Moreover, if $Ric^L_q(W, W) > 0$ for some $q \in \mathcal{L}^\perp$ and all $W \in S_q$ the bounds are $1 \leq b_1(X) \leq 2$ and $0 \leq b_1(X) \leq 1$ respectively.

Proof. Using the Mayer-Vietoris argument and Prop. 3.1 we conclude $b_1(X) \leq b_1(\mathcal{L}^\perp) + 1 \leq \dim X$ if X is compact. In the non-compact case we observed $X \cong \mathcal{L}^\perp \times \mathbb{R}$, i.e., $b_1(X) = b_1(\mathcal{L}^\perp) \leq \dim X - 1$. ∎

Proposition 3.3. *The bounds in Cor. 3.2 are optimal.*

Proof. First, we consider the upper bounds. If (M, g) is a compact Riemannian manifold we derive weakly irreducible Lorentzian metrics on $S^1 \times M \times \mathbb{R}$ and on $S^1 \times M \times S^1$ as follows: If ∂_x is the coordinate field on S^1 define $g^L := 2dxdz + fdz^2 + g$ where ∂_z is the coordinate field of the last factor and $f \in C^\infty(M)$ is suitable. Then $\Xi = TS^1$, $\mathcal{L}^\perp \cong S^1 \times M$ and $\nabla^S|_{\mathcal{L}^\perp}$ is flat if (M, g) is the flat torus.

For the second statement let (M, g) be a compact simply connected Riemannian manifold with strictly positive Ricci curvature. Hence, $Ric^T > 0$ and the upper bounds are optimal.

For the lower bounds let (M, g) be a compact simply connected Calabi-Yau manifold, i.e., $Hol(M, g) = SU(n)$. Consider the total space $\tilde{M}$ of the S^1-bundle given by $0 \neq \alpha \in H^{1,1}_{prim}(M) \cap H^2(M, \mathbb{Z})$. It is shown in [Lär08, Cor. 4.4] that $X := \tilde{M} \times S^1$ and $X := \tilde{M} \times S^1$ admit weakly irreducible Lorentzian metrics g^L such that $Hol(X, g^L) = SU(n) \ltimes \mathbb{R}^{2n}$. In particular, $\mathcal{L}^\perp \cong \tilde{M}$ and $\nabla^S|_{\mathcal{L}^\perp}$ is Ricci flat. The Gysin sequence for the S^1-bundles implies $b_1(\mathcal{L}^\perp) = 0$ since $0 \neq \alpha \in H^2(M, \mathbb{R})$.

Finally, we study the lower bounds if $Ric^T > 0$. Let (M, g) be a compact simply connected Riemannian manifold with strictly positive Ricci curvature and let $\alpha \in H^2(M, \mathbb{Z})$ be a generator. Using the construction in [Lär08] we derive weakly irreducible Lorentzian metrics on $X = \tilde{M} \times S^1$ and on $X = \tilde{M} \times \mathbb{R}$ where $\tilde{M}$ is the total space of the S^1-bundle given by α. Moreover, $Ric^T|_{S \times S} = Ric(M, g)$ and $\mathcal{L}^\perp \cong \tilde{M}$. Hence, $b_1(\mathcal{L}^\perp) = 0$ by the Gysin sequence. ■

We say (X, g^L) satisfies the *strong energy (timelike convergence) condition* at $p \in X$ if $Ric^L_p(W, W) \geq 0$ for any timelike vector $W \in T_pX$. If $\nabla^L_\cdot V = 0$ we have $Ric^L(V, \cdot) = 0$ and $Ric^L(Z, Z) = \sum_k g^L(R^L(Z, Y_k)Y_k, Z)$ as well as $Ric^L(Z, Y_i) = \sum_k g^L(R^S(Z, Y_k)Y_k, Y_i)$.

Remark 3.4. Let (X, g^L, V) be a decent spacetime such that $\nabla^L_\cdot V = 0$ and $p \in X$. If $Ric^L_p(Z, Z) = 0$ and $\sum_k R^S_p(Z, Y_k)Y_k = 0$ then (X, g^L) satisfies the strong energy condition at $p \in X$ if and only if $Ric^L_p(W, W) \geq 0$ for all $W \in \Xi^\perp_p$. ■

4. Screen Holonomy and the Topology of Decent Spacetimes

If there is an integrable realization of the screen bundle the Blumenthal-Hebda decomposition theorem [BH83] immediately implies

Corollary 4.1. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose S is an integrable realization of the screen bundle and $p \in \mathcal{L}^\perp$.*

1. *If $g^R|_{\mathcal{L}^\perp}$ is complete then $\tilde{\mathcal{L}}^\perp = \mathbb{R} \times \tilde{\mathcal{S}}$ where $\tilde{\mathcal{L}}^\perp$ is the universal cover of $\mathcal{L}^\perp$ and $\tilde{\mathcal{S}}$ is the universal cover of a leaf of $S|_{\mathcal{L}^\perp}$.*
2. *If (X, g^L, V, S) is horizontal such that $(L_Z g^L)|_{S \times S} = 0$ and g^R is complete then $\tilde{X} = \mathbb{R}^2 \times \tilde{\mathcal{S}}$ where $\tilde{X}$ is the universal cover of X.*
3. *In both cases, if $Hol^0(\nabla^S) = H_1 \times H_2$ then $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2$ as Riemannian manifolds and $Hol(\tilde{\mathcal{S}}_i) \subset H_i$.*

Proof. Since $g^R|_{\mathcal{L}^\perp}$ is bundle-like for $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ the leaves of $S|_{\mathcal{L}^\perp}$ are totally geodesic in $\mathcal{L}^\perp$ and we can apply the Blumenthal-Hebda theorem.

As we have seen above V and Z induce a 2-dimensional foliation on X if (X, g^L, V, S) is horizontal and g^R is bundle-like for this foliation if $(L_Z g^L)|_{S \times S} = 0$. The Blumenthal-Hebda theorem implies $\tilde{X} = M \times \tilde{\mathcal{S}}$ where M the universal cover of a leaf of the foliation induced by V and Z. Since M is a simply connected parallelizable surface the uniformization theorem implies $M \cong \mathbb{R}^2$.

The last statement follows from the de Rham decomposition theorem since $Hol(\nabla^S|_{\tilde{\mathcal{S}}}) \subset Hol(\nabla^S|_{\tilde{\mathcal{L}}^\perp}) \subset Hol^0(\nabla^S)$. ■

Remark 4.2. If M is a compact simply connected manifold and $X \to M$ is an S^1-bundle whose Euler class is a generator of $H^2(M, \mathbb{Z})$ then the universal cover of X is compact. Using [Lär08] we derive a decent Lorentzian metric on $X \times \mathbb{R}$ which does not admit an integrable realization of the screen bundle. In fact, using the

Milnor-Wood inequality [Woo71] we can construct 4-dimensional decent spacetimes such that $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ does not admit a transverse foliation. ∎

By Cor. 2.7 and [Con74, Prop. 1.6] the foliated manifold $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ admits a transverse G-structure if $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset G$. Note that $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset Hol(\nabla^S)$. The classification of Lorentzian holonomy representations, i.e., representations of $\mathfrak{hol}(X, g^L)$ has been achieved by Leistner in [Lei07] and the hard part is to show that $\mathfrak{hol}(\nabla^S)$ acts as a Riemannian holonomy representation. Galaev [Gal06] constructed real analytic decent spacetimes for all possible representations of $\mathfrak{hol}(\nabla^S)$ for which $\mathfrak{hol}(\nabla^S|_{\mathcal{L}^\perp})$ is trivial for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$. Since $\mathfrak{hol}(\nabla^S)$ has the Borel-Lichnérowicz property (cf. [Lei07, Thm. 2.1]) we have decompositions

$$S_p = E_0 \oplus \ldots \oplus E_\ell \quad \text{and} \quad Hol^0_p(\nabla^S) = H_1 \oplus \ldots \oplus H_\ell$$

where H_j acts irreducibly on E_j for $j \geq 1$. If $\gamma : [0,1] \to X$ is a piecewise smooth curve such that $\gamma(0) = p$ and if τ^S_γ is the parallel displacement w.r.t. ∇^S along γ we define $R_p^{\tau^S_\gamma}(v, w) := {\tau^S_\gamma}^{-1} \circ R^S_{\gamma(1)}(w, v) \circ \tau^S_\gamma$ for $v, w \in S_{\gamma(1)}$. The Ambrose-Singer theorem and $R^S(V, \Xi^\perp) = 0$ imply

$$\mathfrak{hol}_p(\nabla^S|_{\mathcal{L}^\perp}) = \operatorname{span}\{R_p^{\tau^S_\gamma}(\tau^S_\gamma v, \tau^S_\gamma w) : v, w \in S_p, \gamma : [0,1] \to \mathcal{L}^\perp\}.$$

Moreover, each $R_p^{\tau^S_\gamma}(\tau^S_\gamma(\cdot), \tau^S_\gamma(\cdot))$ is an algebraic curvature tensor on S_p. Hence, $\mathfrak{hol}_p(\nabla^S|_{\mathcal{L}^\perp})$ is a Berger algebra in $\mathfrak{so}(S_p)$, i.e., it acts as a Riemannian holonomy representation. Since each subspace E_j is $Hol^0_p(\nabla^S|_{\mathcal{L}^\perp})$-invariant we may consider

$$\mathcal{K}(E_j) := \operatorname{span}\{R_p^{\tau^S_\gamma}(\tau^S_\gamma(\cdot), \tau^S_\gamma(\cdot))|_{E_j \times E_j \times E_j}\}.$$

Suppose $0 \neq \tilde{R} \in \mathcal{K}(E_k)$. Then $(E_k, \tilde{R}, H_k)$ is an irreducible holonomy system and Simons' theorem [Sim62] implies that H_k acts on E_k as a Riemannian holonomy representation.

Lemma 4.3. *Let (X, g^L, V) be an almost decent spacetime and S a realization of the screen bundle. Suppose there is a leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ such that $(\mathcal{L}^\perp, g^R|_{\mathcal{L}^\perp})$ is complete. If $p \in \mathcal{L}^\perp$ and $\mathcal{K}(E_k) = 0$ then $\tilde{\mathcal{L}}^\perp = A \times \mathbb{R}^{\dim E_k}$ where $\tilde{\mathcal{L}}^\perp$ is the universal cover of $\mathcal{L}^\perp$.*

Proof. Consider the foliated manifold $(\tilde{\mathcal{L}}^\perp, \tilde{\mathcal{X}}|_{\mathcal{L}^\perp}, \tilde{g}^R|_{\mathcal{L}^\perp})$ and the lifted connection $\tilde{\nabla}^S|_{\mathcal{L}^\perp}$. Since $\tilde{\mathcal{L}}^\perp$ is simply connected we have $\tilde{\nabla}^S|_{\mathcal{L}^\perp}$-parallel orthonormal sections $Y_1, \ldots, Y_{\dim E_k} \in \Gamma(\tilde{\mathcal{L}}^\perp, \tilde{S})$. An integral curve of any Y_i is a horizontal $\tilde{g}^R|_{\mathcal{L}^\perp}$-geodesic. Hence, each Y_i is a complete vector field on $\tilde{\mathcal{L}}^\perp$. Define $\mathcal{T}^{Y_1} := \operatorname{span}\{Y_1\}^\perp \subset T\tilde{\mathcal{L}}^\perp$. If $W \in \Gamma(U, \mathcal{T}^{Y_1})$ is a local section then $[W, Y_1] \in \tilde{\nabla}^S_W Y_1 - \tilde{\nabla}^S_{Y_1} W + \tilde{\mathcal{X}}|_{\mathcal{L}^\perp} \subset \mathcal{T}^{Y_1}$. Moreover, $\tilde{\nabla}^S_\cdot \mathcal{T}^{Y_1} \subset \mathcal{T}^{Y_1}$. Thus, $\mathcal{T}^{Y_1}$ induces a transversely parallelizable codimension one foliation in $\mathcal{L}^\perp$ and [Con74, Prop. 5.3] implies $\mathcal{L}^\perp = A_{Y_1} \times \mathbb{R}$ where A_{Y_1} is a leaf of $\mathcal{T}^{Y_1}$. For $i \geq 2$ we restrict the vector fields Y_i to A_{Y_1}. As above, we derive a transversely parallelizable codimension one foliation on A_{Y_1} induced by $\mathcal{T}^{Y_2} := \operatorname{span}\{Y_2|_{A_{Y_1}}\}^\perp$ and Y_2 is a complete transverse vector field. Inductively, we have $\tilde{\mathcal{L}}^\perp = A \times \mathbb{R}^{\dim E_k}$ ∎

Theorem 4.4. *Let (X, g^L) be a time-orientable Lorentzian manifold such that $\mathfrak{hol}(X, g^L)$ acts weakly irreducible and reducible. Suppose the associated foliation $\mathcal{X}^\perp$ admits a compact leaf $\mathcal{L}^\perp$ such that $\pi_1(\mathcal{L}^\perp)$ is finite. Then $\mathfrak{hol}(X, g^L)$ belongs to one of the following types where $\mathfrak{g} := \mathfrak{hol}(\nabla^S)$.*

- *Type 1:* $\mathfrak{hol}(X, g^L) = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^{\dim X - 2}$
- *Type 2:* $\mathfrak{hol}(X, g^L) = \mathfrak{g} \ltimes \mathbb{R}^{\dim X - 2}$

- *Type 3:*
$$\mathfrak{hol}(X,g^L)=\left\{\begin{pmatrix}\varphi(A) & w^T & 0\\ 0 & A & -w\\ 0 & 0 & -\varphi(A)\end{pmatrix} : A\in\mathfrak{g},\ w\in\mathbb{R}^q\right\}$$
where $\varphi:\mathfrak{g}\twoheadrightarrow\mathbb{R}$ is an epimorphism satisfying $\varphi|_{[\mathfrak{g},\mathfrak{g}]}=0$.

Moreover, identifying $\mathfrak{g}\subset\mathfrak{so}(\dim X-2)$ there are decompositions
$$\mathbb{R}^{\dim X-2}=F_1\oplus\ldots\oplus F_\ell\quad \text{and}\quad \mathfrak{g}=\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_\ell$$
such that each $\mathfrak{g}_j$ acts trivially on F_i for $i\neq j$ and as an irreducible Riemannian holonomy representation on F_j. In particular, $\mathfrak{g}$ does not act trivially on any subspace of $\mathbb{R}^{\dim X-2}$.

Proof. The universal cover of $\mathcal{L}^\perp$ is compact and Lemma 4.3 implies that $\mathfrak{g}$ does not act trivially on any subspace of $\mathbb{R}^{\dim X-2}$. It is shown in [BBI93] that if $\mathfrak{hol}(X,g^L)$ does not belong to one of the three types then it is given as follows. There is $0<\ell<q$ such that $\mathbb{R}^q=\mathbb{R}^\ell\oplus\mathbb{R}^{q-\ell}$, $\mathfrak{g}\subset\mathfrak{so}(\ell)$ and
$$\mathfrak{hol}(X,g^L)=\left\{\begin{pmatrix}0 & \psi(A)^T & w^T & 0\\ 0 & 0 & 0 & -\psi(A)\\ 0 & 0 & A & -w\\ 0 & 0 & 0 & 0\end{pmatrix} : A\in\mathfrak{g},\ w\in\mathbb{R}^\ell\right\}$$
for some epimorphism $\psi:\mathfrak{g}\twoheadrightarrow\mathbb{R}^{q-\ell}$ satisfying $\psi|_{[\mathfrak{g},\mathfrak{g}]}=0$. Since $\mathfrak{g}$ acts trivially on $\mathbb{R}^{q-\ell}$ we derive a contradiction. ∎

Let A be a global section of some tensor bundle of S and suppose that $A|_{\mathcal{L}^\perp}$ is invariant under the action of $Hol(\nabla^S|_{\mathcal{L}^\perp})$ for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$. Then A is invariant under the action of $Hol(\nabla^S)$ if and only if $\nabla^S_Z A=0$. We remind that $d(g^L(Z,\cdot))|_{\mathcal{L}^\perp}$ induces the Euler class of $\mathcal{L}^\perp$ if (X,g^L,V,S) is almost horizontal.

Lemma 4.5. *Let (X,g^L,V) be an almost decent spacetime and S a realization of the screen bundle. If $J\in\Gamma(X,O(S))$ with $J^2=-id_S$ then $\nabla^S J=0$ if and only if $\nabla^S|_{\mathcal{L}^\perp}J|_{\mathcal{L}^\perp}=0$ for any leaf $\mathcal{L}^\perp$ of $\mathcal{X}^\perp$ and*
$$\begin{aligned}0&=d(g^L(Z,\cdot))(JY_1,Y_2)+d(g^L(Z,\cdot))(Y_1,JY_2)\\&\quad+g^L((L_ZJ)(Y_1),Y_2)-g^L((L_ZJ)(Y_2),Y_1).\end{aligned}$$

Proof. Define the extension $J\in\Gamma(X,End(TX))$ by $J(V)=J(Z)=0$ and let $\omega(\cdot,\cdot):=g^L(J(\cdot),\cdot)\in\Lambda^2T^*X$. Since $(L_ZJ)(Y)=[Z,JY]-J([Z,Y])$ we compute for $Y_1,Y_2\in\Gamma(U,S)$
$$\begin{aligned}g^L((\nabla^S_ZJ)(Y_1),Y_2)&=g^L(\nabla^S_Z(JY_1),Y_2)-g^L(J\nabla^S_ZY_1,Y_2)\\&=g^L([Z,JY_1],Y_2)+g^L(\nabla^L_{JY_1}Z,Y_2)\\&\qquad+g^L([Z,Y_1],JY_2)+g^L(\nabla^L_{Y_1}Z,JY_2)\\&=g^L((L_ZJ)(Y_1),Y_2)+g^L(\nabla^L_{JY_1}Z,Y_2)+g^L(\nabla^L_{Y_1}Z,JY_2),\ \text{i.e.,}\end{aligned}$$
$g^L((\nabla^S_ZJ)(Y_1),Y_2)-g^L((\nabla^S_ZJ)(Y_2),Y_1)=g^L((L_ZJ)(Y_1),Y_2)-g^L((L_ZJ)(Y_2),Y_1)+d(g^L(Z,\cdot))(JY_1,Y_2)+d(g^L(Z,\cdot))(Y_1,JY_2)$. We conclude the statement since $\nabla^S_\cdot\omega$ is a 2-form on S and $\nabla^S_Z\omega(Y_1,Y_2)=g^L((\nabla^S_ZJ)(Y_1),Y_2)$. ∎

In order to estimate the higher Betti numbers we have to use the dual basic cohomology in the Gysin sequence of the flow if the basic cohomology does not satisfy Poincaré duality. This is the case if and only if the Riemannian foliation $(\mathcal{L}^\perp,\mathcal{X}|_{\mathcal{L}^\perp})$ is not taut [HR10]. Here we say $(\mathcal{L}^\perp,\mathcal{X}|_{\mathcal{L}^\perp})$ is taut if $H_B^{\dim\mathcal{L}^\perp-1}(\mathcal{X}|_{\mathcal{L}^\perp})\neq0$ which is equivalent to the vanishing of the Álvarez-class $[\kappa_{\tilde g^B}]\in H^1_B(\mathcal{X}|_{\mathcal{L}^\perp})$.

By Cor. 2.7 and [Con74, Prop. 1.6] the condition $\nabla^S|_{\mathcal{L}^\perp} J|_{\mathcal{L}^\perp} = 0$ means that $J|_{\mathcal{L}^\perp}$ induces a Kähler foliation on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp}, g^R|_{\mathcal{L}^\perp})$. In particular, basic Dolbeault cohomology is defined on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ [EKA90]. Suppose that $\mathcal{L}^\perp$ is compact such that $Ric^L(W,W) \geq 0$ for all $W \in T\mathcal{L}^\perp$ and $Hol(\nabla^S|_{\mathcal{L}^\perp})$ is irreducible. As in Prop. 3.1 we conclude $\dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) = 0$ since there is no $Hol(\nabla^S|_{\mathcal{L}^\perp})$-invariant vector. Hence, $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut.

Lemma 4.6. *Let (X, g^L, V) be an almost decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. If $Ric^L|_{T\mathcal{L}^\perp \times T\mathcal{L}^\perp} = 0$ and $Hol(\nabla^S|_{\mathcal{L}^\perp}) \subset U(n)$ then any basic $(p,0)$-form ψ on $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is closed if and only if $\nabla^S|_{\mathcal{L}^\perp}\psi = 0$.*

Proof. One part of the proof is implied by $d\psi = \sum_{i=1}^{\dim S} e^i \wedge \nabla^S_{e_i}\psi$.

Since ψ is a $(p,0)$-form we have $\bar{\delta}_b\psi = 0$ and $d\psi = 0$ implies $\bar{\partial}\psi = 0$, i.e., $\Delta_b^{\bar{\partial}}\psi = 0$. Thus, $\Delta_b\psi = 0$ by the transverse Kähler identities [EKA90] and we have $d\psi = \delta_b\psi = 0$ implying $\int_{\mathcal{L}^\perp} \langle \Delta_\kappa\psi, \psi\rangle = \frac{1}{4}|\kappa|^2|\psi|^2$. By the Weitzenböck formula

$$0 = \int_{\mathcal{L}^\perp} |\nabla^T\psi|^2 + \int_{\mathcal{L}^\perp} \langle \sum\nolimits_{i,j} e^j \wedge e_i \lrcorner R^T(e_i, e_j)\psi, \psi\rangle.$$

However, R^T beeing the curvature of $\nabla^S|_{\mathcal{L}^\perp}$ has the same symmetries as the curvature tensor of a Kähler manifold. Using the computation in [Joy00, Prop. 6.2.4] we conclude $\sum_{i,j} e^j \wedge e_i \lrcorner R^T(e_i, e_j)\psi = 0$, i.e., $0 = \int_{\mathcal{L}^\perp} |\nabla^T\psi|^2$. ■

Proposition 4.7. *Let (X, g^L, V) be a decent spacetime and $\mathcal{L}^\perp$ a compact leaf of $\mathcal{X}^\perp$. Suppose $Hol(\nabla^S|_{\mathcal{L}^\perp})$ is irreducible and $Ric^L(W,W) \geq 0$ for all $W \in T\mathcal{L}^\perp$.*

1. *If X is compact then $b_1(X) \in \{1,2\}$ and $b_2(X) \leq \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$.*
2. *If X is non-compact and all leaves of $\mathcal{X}^\perp$ are compact then $b_1(X) \in \{0,1\}$ and $b_2(X) \in \{\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) - 1, \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})\}$.*

Moreover, if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = SU(n)$ with $n \geq 3$ we can replace $H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp})$ and if $Hol(\nabla^S|_{\mathcal{L}^\perp}) = Sp(n)$ with $n \geq 1$ we can replace $\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})$ by $\dim H^{1,1}_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 2$.

Proof. Since $\dim H^1_B(\mathcal{X}|_{\mathcal{L}^\perp}) = 0$ and $(\mathcal{L}^\perp, \mathcal{X}|_{\mathcal{L}^\perp})$ is taut we derive the bounds for $b_1(X)$ and the Gysin sequence implies $\mathbb{R} \xrightarrow{[\cdot\wedge\mathbf{e}]} H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) \longrightarrow H^2(\mathcal{L}^\perp, \mathbb{R}) \longrightarrow 0$, i.e., $b_2(\mathcal{L}^\perp) \in \{\dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) - 1, \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp})\}$. If X is compact the Mayer-Vietoris argument implies $H_2(\mathcal{L}^\perp) \xrightarrow{Id - F^2_*} H_2(\mathcal{L}^\perp) \xrightarrow{\iota_*} H_2(X) \longrightarrow H_1(\mathcal{L}^\perp) \xrightarrow{Id - F^1_*} H_1(\mathcal{L}^\perp)$. Hence, $b_2(X) = b_1(\mathcal{L}^\perp) + \dim \mathrm{Eig}_1(F^2_*)$, where $\mathrm{Eig}_1(F^2_*)$ is the eigenspace of F^2_* w.r.t. the eigenvalue 1, i.e., $b_2(X) \leq b_1(\mathcal{L}^\perp) + b_2(\mathcal{L}^\perp) \leq \dim H^2_B(\mathcal{X}|_{\mathcal{L}^\perp}) + 1$. The last statements follow from Lemma 4.6. ■

Proposition 4.8. *Let (X, g^L, V, S) be a horizontal spacetime and $\mathcal{L}^\perp$ a leaf of $\mathcal{X}^\perp$. Suppose $d(g^L(Z,\cdot))|_{\mathcal{L}^\perp} \in \Lambda^{1,1}_B\mathcal{X}|_{\mathcal{L}^\perp}$ for some ∇^S-parallel almost Hermitian structure J on S. If $Z \in \Gamma(X, TX)$ is complete then there exists a complex structure on the universal cover of X.*

Proof. By Cor. 2.4 the universal cover of X is diffeomorphic to $\tilde{X} := \mathcal{L}^\perp \times \mathbb{R}^+$. We write r for the coordinate on $\mathbb{R}^+$ and $\eta := g^L(Z,\cdot)|_{T\mathcal{L}^\perp}$. If $\Phi \in \mathrm{End}(T\mathcal{L}^\perp)$ is given by $\Phi(w \in S_p) := J(w)$ and $\Phi(V) := 0$ then $(V, \eta, \Phi, g^R|_{\mathcal{L}^\perp})$ defines an almost contact metric structure on $\mathcal{L}^\perp$. On $\tilde{X}$ we define the cone metric $g^C := dr^r + r^2 g^R|_{\mathcal{L}^\perp}$ and the section $I \in \mathrm{End}(T\tilde{X})$ by

$$IY := \begin{cases} JY & \text{if } Y \in S_p, \\ r\partial_r & \text{if } Y = V, \\ -V & \text{if } Y = r\partial_r. \end{cases}$$

Hence, we derive an almost Hermitian manifold $(\tilde{X}, I, g^C)$. By [BG08, Thm. 6.5.9] I is integrable once we prove[7] that $N_\Phi = -V \otimes d\eta$ where $N_\Phi(Y_1, Y_2) := [\Phi Y_1, \Phi Y_2] + \Phi^2[Y_1, Y_2] - \Phi[Y_1, \Phi Y_2] - \Phi[\Phi Y_1, Y_2]$ for $Y_\cdot \in T\mathcal{L}^\perp$. Let $Y_1 \in S$ and $Y_2 = V$. Since (X, g^L, V, S) is horizontal we have $[Y_1, V] = -\nabla^S_V Y_1$. Thus, $\Phi V = 0$ and $J \circ \nabla^S = \nabla^S \circ J$ implies

$$\begin{aligned} N_\Phi(Y_1, V) &= \Phi^2[Y_1, V] - \Phi[\Phi Y_1, V] = -\Phi^2(\nabla^S_V Y_1) + \Phi(\nabla^S_V \Phi Y_1) \\ &= -J^2(\nabla^S_V Y_1) + J(\nabla^S_V J Y_1) = 0. \end{aligned}$$

The same way we compute $g^L(N_\Phi(Y_1, Y_2), Y_3) = 0$ if $Y_1, Y_2, Y_3 \in S$. For $Y_1, Y_2 \in S$ we have

$$\begin{aligned} g^L(N_\Phi(Y_1, Y_2), Z) &= g^L([\Phi Y_1, \Phi Y_2], Z) = g^L([JY_1, JY_2], Z) \\ &= -g^L(\nabla^L_{JY_1} Z, JY_2) + g^L(\nabla^L_{JY_2} Z, JY_1) \\ &= -d(g^L(Z, \cdot))|_{\mathcal{L}^\perp}(JY_1, JY_2). \end{aligned}$$

We conclude $g^L(N_\Phi(Y_1, Y_2), Z) = -d\eta(Y_1, Y_2)$ since $d(g^L(Z, \cdot))|_{\mathcal{L}^\perp} \in \Lambda^{1,1}_B \mathcal{X}|_{\mathcal{L}^\perp}$ and $d\eta = d(g^L(Z, \cdot))|_{\mathcal{L}^\perp}$. ■

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Humboldt-Universität Berlin, Institut für Mathematik R. 1.305, Rudower Chaussee 25, 12489 Berlin

E-mail address: laerz@math.hu-berlin.de